COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA **15** (2) (1995) 159–165

BOUNDING THE NUMBER OF BASES OF A MATROID

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Received March 3, 1993
Revised December 15, 1993

Let $b(M)$ and $c(M)$, respectively, be the number of bases and circuits of a matroid M. For any given minor closed class $\mathcal{M}$ of matroids, the following two questions are investigated in this paper. (1) When is there a polynomial function $p(x)$ such that $b(M) \le p(c(M)|E(M)|)$ for every matroid M in $\mathcal{M}$? (2) When is there a polynomial function $p(x)$ such that $b(M) \le p(|E(M)|)$ for every matroid M in $\mathcal{M}$? Let us denote by M_n the direct sum of n copies of $U_{1,2}$. We prove that the answer to the first question is affirmative if and only if some M_n is not in $\mathcal{M}$. Furthermore, if all the members of $\mathcal{M}$ are representable over a fixed finite field, then we prove that the answer to the second question is affirmative if and only if, also, some M_n is not in $\mathcal{M}$.

1. Introduction

Let $b(M)$ and $c(M)$, respectively, be the number of bases and circuits of a matroid M. For any given minor closed class $\mathcal{M}$ of matroids, we shall investigate the following two questions in this paper. (1) When is there a polynomial function $p(x)$ such that $b(M) \le p(c(M)|E(M)|)$ for every matroid M in $\mathcal{M}$? (2) When is there a polynomial function $p(x)$ such that $b(M) \le p(|E(M)|)$ for every matroid M in $\mathcal{M}$?

For each positive integer n, let M_n be the direct sum of n copies of $U_{1,2}$. Then, for every matroid M, we define $a(M)$ to be the smallest positive integer n so that M does not have M_n as a minor. To answer the above two questions, we prove the following two theorems which are the main results of this paper.

Theorem 1. *Let M be a matroid with $c(M) > 0$ and let $h(M)$ denote the size of the largest circuit of M. Let $f(a) = 22(a+1)^3 \binom{2a-1}{a}^2$ for all positive integers a. Then $b(M) \le f(a(M))(c(M)h(M))^{f(a(M))}$.*

Theorem 2. *There exists a function $g(x,y)$ with the following property. Let M be an F-representable matroid, where F is a finite field, and let E_0 be the set of elements of M that are not loops or coloops. If $E_0 \ne \emptyset$, then $b(M) \le g(a(M),|F|)|E_0|^{3g(a(M),|F|)/2}$.*

The motivation of this research is the study of *polynomially tied* parameters. Two matroid-parameters α and β are *polynomially tied* for a class $\mathcal{M}$ of matroids if

Mathematics Subject Classification (1991): 05 B 35, 05 C 55, 05 C 10 there are two polynomial functions $p(x)$ and $q(x)$ such that $\alpha(M) \le p(\beta(M))$ and $\beta(M) \le q(\alpha(M))$ for all members M of $\mathcal{M}$.

For a general matroid M, $b(M)$ and $c(M)$ may not be related at all (consider $U_{n,n+1}$ and $U_{0,n}$ for positive integers n). Thus we introduce the following notation. Let $\bar{b}(M) = \max\{b(M), |E(M)|\}$ and $\bar{c}(M) = \max\{c(M), |E(M)|\}$. Then, since $c(M) \le r(M^*)b(M)$ (this inequality can be easily verified), we have $\bar{c}(M) \le (\bar{b}(M))^2$. Therefore, question (1) can be rephrased as "when are $\bar{b}$ and $\bar{c}$ polynomially tied?". Notice that, if $\bar{b}$ and $\bar{c}$ are polynomially tied for a minor closed class $\mathcal{M}$ of matroids, then some M_n must not be contained in $\mathcal{M}$. Thus we deduce from Theorem 1 that $\bar{b}$ and $\bar{c}$ are polynomially tied for a minor closed class $\mathcal{M}$ of matroids if and only if some M_n is not contained in $\mathcal{M}$. Consequently, the answer to question (1) is affirmative if and only if some M_n is not in $\mathcal{M}$. There is a another corollary of Theorem 1 worth mentioning. It is well known [4] that, if a matroid M is cosimple, then $|E(M)| \le c(M)$. Thus we have the following result.

Corollary. *Let M be a matroid for which no two elements are in series. If $c(M) > 0$, then $b(M) \le f(a(M))(c(M))^{2f(a(M))}$.*

Let $E_0(M)$ be the set of elements of M that are neither loops nor coloops. Then one may also compare $b(M)$ with $|E_0(M)|$. It is proved by Murty (in [4], page 301) that $|E_0(M)| \le b(M)$ for every matroid M. Thus question (2) can be rephrased as "for which minor closed class $\mathcal{M}$ of matroids are $b(M)$ and $|E_0(M)|$ polynomially tied?". An obvious necessary condition for $\mathcal{M}$ to be such a class is that $\bar{b}$ and $\bar{c}$ are polynomially tied for $\mathcal{M}$. But, by considering the class of all uniform matroids, it is not difficult to see that this condition is not sufficient. However, from Theorem 2 we conclude that, if all the members of $\mathcal{M}$ are representable over a fixed finite field, then the above obvious necessary condition is indeed also a sufficient condition. Clearly, this conclusion partially answers question (2). Now a natural question is: is it true that $b(M)$ can be bounded from above by a polynomial function of $|E_0(M)|$ for which the order depends only on $a(M)$ and the size of the largest $U_{n,2n}$ minor of M? An affirmative answer to this question would answer question (2) completely.

If graphs are considered instead of matroids, then there is a stronger result[1]. For graphs, the minor relation can be replaced by the weaker topological minor relation. Moreover, by a completely different approach, a function which is much better than the one given in the proof of Theorem 2 is derived.

The rest of this paper is organized as follows. Theorem 1 is proved in Section 2 and Theorem 2 is proved in Section 3. Also in Section 3, we discuss the structure of a matroid that does not have a big M_n minor.

2. Bounding $b(M)$ in terms of $\bar{c}(M)$

To prove Theorem 1, we need some preparations. A *hypergraph* H is a pair (V, E) such that V is a finite set and E is a collection of subsets of V. The members of V and E are called *vertices* and *edges* of H, respectively. A subset U of V is *stable* if no two vertices in U are contained in the same edge. A subset F of E is a *cover* if V is the union of the edges in F. We shall denote the size of the largest stable set of H by $\alpha(H)$ and the size of the smallest cover of H by $\rho(H)$, where $\rho(H)$ is defined to be ∞ if H does not have a cover. A subset U of V is *d-faithful*, where d is a positive integer, if $|U| \geq d$ and, for every subset X of U with $|X| = d$, there exists an edge A in E such that $U \cap A = X$. The size of the largest d-faithful set of H shall be denoted by $\lambda_d(H)$. If H has no d-faithful set, then $\lambda_d(H)$ is defined to be d. In the rest of the paper, if H is understood, then we denote $\alpha(H)$, $\rho(H)$ and $\lambda_d(H)$ by α, ρ and λ_d, respectively. The following lemma [3] provides an upper bound of ρ in terms of α and λ_2.

Lemma 2.1. *If $\rho(H)$ is finite, then* $\rho \leq 11\lambda_2^2(\lambda_2 + \alpha + 3)\binom{\lambda_2 + \alpha}{\alpha}^2$.

As a corollary, we have the next lemma which provides an upper bound of ρ in terms of λ_1.

Lemma 2.2. *If $\rho(H)$ is finite, then* $\rho \leq 22(\lambda_1 + 2)^3\binom{2\lambda_1 + 1}{\lambda_1}^2$.

Proof. We need only show, by Lemma 2.1, that $\alpha \leq \lambda_1$ and $\lambda_2 - 1 \leq \lambda_1$. Since $\rho(H)$ is finite, each vertex of H is contained in at least one edge of H. It follows that all nonempty stable sets of H are 1-faithful and thus the first inequality holds. To prove the second inequality, clearly, we may assume that H does have 2-faithful sets. Let U be a 2-faithful set of H with $|U| = \lambda_2(H)$. Let u be a vertex in U. Then, for every vertex v in $U - \{u\}$, there is an edge A with $A \cap U = \{u, v\}$. It follows that $U - \{u\}$ is 1-faithful and so the second inequality holds as well. ∎

Let M be a matroid on E and let B be a basis of M. For every element e in $E - B$, let $C(e,B)$ be the fundamental circuit of e with respect to B. Now the hypergraph $H(M,B)$ is defined to be $(B, \{C(e,B) - \{e\} : e \in E - B\})$.

Lemma 2.3. *Let M be a matroid for which there is a circuit of size at least two. Then $\lambda_1(H(M,B)) < a(M)$ for every basis B of M.*

Proof. Let B be a basis of M. Since M has a circuit of size at least two, $H(M,B)$ must have 1-faithful sets. Let U be a 1-faithful set of $H(M,B)$ with $|U| = \lambda_1$. For each u in U, let z_u be an element in $E - B$ such that $U \cap C(z_u, B) = \{u\}$. Let $Z = \{z_u : u \in U\}$. Then it is easy to see that $M \backslash (E - B - Z)/(B - U)$ is isomorphic to M_{λ_1} and thus the result follows. ∎

Proof of Theorem 1. Without loss of generality, we assume that M has no coloop and that $h(M) \geq 2$. Let us consider pairs (C, x), where C is a circuit of M and x is an element in C. Clearly, there are at most $c(M)h(M)$ such pairs. Since M has no coloop and $c(M)$ is not zero, it follows that $\rho(H(M,B))$ is finite for every basis B of M. Then, from Lemma 2.2 and Lemma 2.3 we deduce that $\rho(H(M,B)) \leq f(a(M))$ for every basis B of M. As a consequence, for every basis B of M, there are i pairs (C_1, x_1), ..., (C_i, x_i), where $i \leq f(a(M))$, such that B is the union of $C_1 - \{x_1\}$, ..., $C_i - \{x_i\}$. Therefore,

$$b(M) \leq \sum_{i=1}^{f(a(M))} \binom{c(M)h(M)}{i} \leq f(a(M))(c(M)h(M))^{f(a(M))}$$

as required. ∎

3. Bounding $b(M)$ in terms of $|E_0(M)|$

A matroid is *diagonal* if it has a basis which is also a cobasis. For every matroid M, we denote by $d(M)$ the largest integer d such that M has a diagonal matroid of rank d as a minor.

Lemma 3.1. *If $b(M)\neq 1$, then $b(M)\leq d(M)|E(M)|^{3d(M)/2}$.*

Proof. Since $b(M)\neq 1$, there exists an element e of M which is neither a loop nor a coloop. Choose a subset Z of $E(M)$ such that
(i) Z is independent in both M and M^*, and
(ii) subject to (i), the cardinality d of Z is maximum.

The existence of Z that satisfies (i) is clear since we may take $\{e\}$ to be Z. Consequently, d is not zero. Extend Z, respectively, into a basis B and a cobasis B^*. Then it is clear that Z is both a basis and a cobasis of $M\backslash(B^*-Z)/(B-Z)$. It follows that $M\backslash(B^*-Z)/(B-Z)$ is a diagonal matroid of rank d and thus $d\leq d(M)$.

Let us denote $E(M)$ and $r(M)$ by E and r respectively. Then from the theorem of Edmonds on common independent sets of two matroids (see [4], page 130), we conclude that there exists a partition (X,Y) of E with $r(X)+r^*(Y)=d$. Without loss of generality, let us assume that $r(X)\leq r^*(Y)$. Since $|B\cap X|\leq r(X)$ for all bases B of M, it follows that

$$b(M)\leq \sum_{i=0}^{r(X)}\binom{|X|}{i}\binom{|Y|}{r-i}=\sum_{i=0}^{r(X)}\binom{|X|}{i}\binom{|Y|}{|Y|-r+i}$$

$$\leq \sum_{i=0}^{r(X)}\binom{|E|}{i}\binom{|E|}{|Y|-r+i}\leq \sum_{i=0}^{r(X)}|E|^i|E|^{|Y|-r+i}\leq (r(X)+1)|E|^{|Y|-r+2r(X)}.$$

Since $r^*(Y)\geq d/2>0$, we must have $r(X)+1\leq r(X)+r^*(Y)=d$. Also, since $|Y|\leq r(Y)+r^*(Y)$, it follows that $|Y|-r+2r(X)\leq r^*(Y)+2r(X)\leq 3d/2$. Therefore, $b(M)\leq d|E|^{3d/2}\leq d(M)|E|^{3d(M)/2}$ as required. ∎

To have an upper bound of $d(M)$ in terms of $a(M)$, we need another two lemmas. Let s and t be positive integers and let $R(s;t)$ (the Ramsey number) denote the smallest integer r such that, for every t-coloring of the edges of a complete graph K_r on r vertices, there exists an induced subgraph of K_r on s vertices for which all the edges have the same color. Let F be a finite set of cardinality at least two and let A be an $m\times n$ matrix such that all its entries are elements of F. If no two distinct columns of A are equal, then we say that A is *simple.* We shall say that A is *weakly diagonal* if $m=n$ and there are elements f_0, f_1 and f_2 in F such that, $f_0\neq f_1$, $A_{i,i}=f_0$ for all i, $A_{i,j}=f_1$ for all $i<j$, and $A_{i,j}=f_2$ for all $i>j$. Our next lemma says that every big simple matrix must have a big weakly diagonal submatrix.

Lemma 3.2. *Let w be a positive integer and let F, A, m, and n be as given in the above. If $n\geq 1+|F|^{|F|^2R(w;|F|)}$, then the rows and columns of A can be permuted so that the resulting matrix has a $w\times w$ weakly diagonal submatrix.*

Proof. We first make the following observation.

(∗) *Let B be an $m\times n'$ submatrix of a permutation of A. Suppose that $n'\geq 1+|F|^t$, where t is a positive integer. Then there are indices i and $j_0, j_1, \ldots, j_{n''}$, where $n''\geq 1+|F|^{t-1}$, such that $B_{i,j_1}=B_{i,j_2}=\ldots=B_{i,j_{n''}}$ and $B_{i,j_0}\neq B_{i,j_1}$.*

To verify this observation, first notice that, since A, and hence B, is simple, there is an index i such that $|\{B_{i,j}: j=1,\ldots,n'\}|\geq 2$. On the other hand, since $n'\geq 1+|F|^t$ and $|\{B_{i,j}: j=1,\ldots,n'\}|\leq |F|$,we deduce that there are indices $j_1,\ldots,j_{n''}$ such that $B_{i,j_1}=\ldots=B_{i,j_{n''}}$ and $n''\geq 1+|F|^{t-1}$. Finally, from the choice of i, the existence of j_0 follows obviously.

Let $r=R(w;|F|)$ and $s=r|F|^2$. Then let us apply observation (∗) to A. It follows that there are indices i and $j_0, j_1, \ldots, j_{n_1}$, where $n_1\geq 1+|F|^{s-1}$, such that $A_{i,j_1}=\ldots=A_{i,j_{n_1}}$ and $A_{i,j_0}\neq A_{i,j_1}$. By permuting the rows and columns of A, we may assume that $i=j_0=1$ and $j_k=k+1$ for $k=1,\ldots,n_1$. Now let B be the submatrix of A that consists of columns indexed by $2,3,\ldots,n_1+1$, and then apply observation (∗) to B. By permuting the rows and columns of B, and hence of A, we may assume that $A_{2,2}\neq A_{2,3}$ and $A_{2,3}=A_{2,4}=\ldots=A_{2,n_2+2}$, where $n_2\geq 1+|F|^{s-2}$. Clearly, if we repeat this process $s-1$ time, then A can be permuted so that $A_{i,i}\neq A_{i,i+1}$ and $A_{i,i+1}=A_{i,i+2}=\ldots=A_{i,s}$ for all $i=1,\ldots,s-1$. Since $s=r|F|^2$ and $|\{(A_{i,i}, A_{i,i+1}): i=1,\ldots,s-1\}|<|F|^2$, it follows that we can permute the rows and columns of A again so that $A_{1,1}=\ldots=A_{r,r}$, $A_{1,1}\neq A_{1,2}$, and $A_{i,j}=A_{1,2}$ for $1\leq i<j\leq r$.

Now consider an $|F|$-coloring of a complete graph G on $\{1,\ldots,r\}$ such that the edge (i,j), where $i>j$, is colored by $A_{i,j}$. From the definition of $R(w;|F|)$ we conclude that there is an induced subgraph of G on $\{i_1,\ldots,i_w\}$ such that all its edges have the same color. Obviously, the submatrix of A on rows $i_1,\ldots,i_w$ and columns $i_1,\ldots,i_w$ is weakly diagonal and thus the proof of Lemma 3.2 is completed. ∎

Remark. Lemma 3.2 was first proved in [2].

Lemma 3.3. *Let F be a finite field and let M be an F-representable matroid. Then, for every cobasis B^* of M, the rank of B^* is at most $|F|^{|F|^2R(3a(M);|F|)}$.*

Proof. Suppose there is a cobasis B^* of M with rank exceeds $|F|^{|F|^2R(3a;|F|)}$, where $a=a(M)$. Let $B=E(M)-B^*$. Then B is a basis of M. Let the matrix (I,A) be a standard representation of M, over F,with respect to the basis B. From the choice of B^* it follows that A has at least $1+|F|^{|F|^2R(3a;|F|)}$ columns for which no two of them are equal. By applying Lemma 3.2 to the submatrix of A that consists of these columns we conclude that a permutation of A has a $3a\times 3a$ weakly diagonal submatrix K. Let us assume, for i, $j=1$, $\ldots$, $3a$, that the ith row and the jth columns of K are indexed respectively by x_i and y_j,where x_i is in B and y_j is in B^*. Let $X=\{x_1,\ldots,x_{3a}\}$,$Y=\{y_1,\ldots,y_{3a}\}$, and $N=M\backslash(B^*-Y)/(B-X)$. We shall deduce that N,and hence M, has a minor isomorphic to M_a. Clearly, it contradicts the definition of $a(M)$ and this contradiction proves the Lemma.

We first consider the case when $K_{1,3a}\neq K_{3a,1}$. Let $X'=\{x_{3i}: i=1,\ldots,a\}$ and $Y'=\{y_{3i}: i=1,\ldots,a\}$. Then it is straightforward to verify that $N\backslash(X'\cup Y')/(X-X')$ is isomorphic to M_a. Thus, we are done. Next, we consider the case when $K_{1,3a}=K_{3a,1}=0$. In this case, it is easy to see that N itself is isomorphic to M_{3a} and hence N has a minor isomorphic to M_a, we are done again. Finally, we consider the case when $K_{1,3a} = K_{3a,1} \neq 0$. By pivoting the matrix K on $K_{3a,1}$ it is not difficult to see that $N\backslash\{x_{3n}, y_{3n}\}/\{x_1, y_1\}$ is isomorphic to M_{3a-2}. From the definition of $a(M)$ we know that a is a positive integer. It follows that, once more, N has a minor isomorphic to M_a as required. ∎

Proof of Theorem 2. Let $g(s,t) = t^{t^2 R(3s;t)}$. We shall prove that $g(s,t)$ has the required property. Clearly, we may assume that $E(M) = E_0 \neq \emptyset$. Let N be a minor of M so that N is diagonal with rank $d(M)$. Choose a basis X of N such that it is also a basis of N^*. Then extend X into a cobasis Y of M. From Lemma 3.3 we deduce that

$$d(M) = |X| = r(X) \leq r(Y) \leq g(a(M), |F|).$$

Thus Theorem 2 follows obviously from Lemma 3.1. ∎

The main results of this section can be summarized as follows.

Theorem 3. *Let $\mathcal{M}$ be a minor-closed class of F-representable matroids, where F is a finite field. Then the following are equivalent.*

(1) *b and $|E_0|$ are polynomially tied for $\mathcal{M}$;*

(2) *Some M_n is not in $\mathcal{M}$;*

(3) *There is a number r such that, for all matroids M in $\mathcal{M}$, we have $r(B^*) \leq r$ and $r^*(B) \leq r$ for all bases B and cobases B^* of M;*

(4) *There is a number r such that, for all matroids M in $\mathcal{M}$, there exists either a basis B of M with $r^*(B) \leq r$ or a basis B^* of M^* with $r(B^*) \leq r$.*

Proof. The implications $(3) \Rightarrow (4) \Rightarrow (2)$ and $(1) \Rightarrow (2)$ are trivial. The implications $(2) \Rightarrow (1)$ and $(2) \Rightarrow (3)$ follow from Theorem 2 and Lemma 3.3, respectively. ∎

Remark. Let $\mathcal{M}$ be a minor-closed class of F-representable matroids, where F is a finite field. Suppose one is interested in finding an optimal basis, with respect to certain criteria, for every matroid in $\mathcal{M}$. One algorithm to solve this problem can be described as follows. For every input matroid M in $\mathcal{M}$, list all bases of M and then find the best one. In general, this algorithm is not very efficient. Here, Theorem 3 can be interpreted as a characterization of $\mathcal{M}$ for which the algorithm does run in polynomial time.

Acknowledgement. The author thanks James Oxley and Paul Seymour for many helpful discussions.

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